JOURNAL OF APPLIED POLYMER SCIENCE VOL. 11, PP. 925-947 (1967)

Fatigue Under Complicated Load Histories. Law of Cumulative Damage

DUSAN C. PREVORSEK and MICHAEL L. BROOKS, *Central Research Laboratory, Allied Chemical Corporation, Morristown, New Jersey 07960*

Synopsis

The applicability of Miner's law of cumulative damage to predict lifetimes in experiments involving complex load histories is examined. Lifetimes estimated by $\Sigma\, t_i/(t_b)_i = 1$ are compared with those calculated by the expression for the time to rupture derived by Prevorsek and Lyons assuming that the time to rupture can be approximated by the time to form an unstable crack. For experiments in which the loading conditions became increasingly severe with time, lifetimes predicted by Miner's law are longer than those calculated from the rate of crack propagation, the opposite being found for experiments in which the loading conditions become decreasingly severe with time. Experimental data on hand are in agreement with these findings. Effects of changes in the structural parameters ρ, E, and ΔF^* and of variations in the experimental conditions on the accuracy of the lifetime estimates are discussed.

INTRODUCTION

Fatigue under a complex loading history represents the most frequent source of failure in end use applications. Laboratory experiments and fatigue theories are, on the other hand, primarily concerned with fatigue phenomena under cyclic loading at constant frequency and constant stress or strain levels. As a result of the increasing number of critical fatigue examples encountered in modern structures and engineering apparatus, there is a rapidly growing need to perform complementary fatigue experiments involving more complex loading histories. Nevertheless, these experiments must be amenable to some more or less rigorous physical analysis. The objective of such experimental and theoretical studies is to establish and explain the superposition rules which govern the rate of failure at some complex stress history. These rules, when established with sufficient accuracy would allow one to solve one of the basic problems of engineering design, namely, how to predict the endurance of materials in actual performance from the results obtained in standard fatigue testing.

One type of study intended to narrow the gap between the information obtained in laboratory experiments and the prediction of performance in practical application is cumulative damage testing. In this test, the specimens are subjected to various levels of stress in sequence which may be interrupted with periods of rest during which the load is completely removed.

The simplest cumulative damage experiment involves the stressing of the specimen to a predetermined stress σ_1, which is applied for a time t_1. At time t_1 the stress is removed, and the specimen is allowed to rest for a time $(t_1)_{\text{rest}}$. At time $t_1 + (t_1)_{\text{rest}}$, the stress σ_1 is applied again for a time t_2 and removed at time $t_1 + t_2 + (t_1)_{\text{rest}}$. After the specimen is allowed to rest for a time $(t_2)_{\text{rest}}$, the stress is applied again. This procedure is continued until the specimen fails. The fact that the sum of times of load application $\Sigma\, t_1$ usually varies little from the time to rupture determined in a separate experiment where σ_1 is applied without interruptions is interpreted as indicating that the internal damage caused by loading of the specimens is not repaired when the stress is released.

In a more complex but frequently used cumulative damage experiment, the stress σ_1 is applied for a time t_1, then the stress is changed to a higher or lower stress σ_2 and applied for a time t_2; at time $t_1 + t_2$ the stress may be changed to a new stress σ_3 which can be equal to or different from σ_1, etc. If $\sigma_1 < \sigma_2 < \sigma_3$, then the specimen will break sooner than it would have broken if subjected solely to σ_1, but not as soon as if only σ_3 had been applied. It has been proposed that the time to failure in such an experiment can be estimated by assuming that the specimen fails at a time at which the sum of the individual fractional lifetimes that have been expended equals unity. The underlying notion of this hypothesis is that if the specimen is subjected to certain fatiguing conditions, the permanent damage sustained can be expressed as the ratio of the time t_1 the specimen had been subjected to a given stress σ_1, to the time to failure $(t_b)_1$ under the same particular conditions. If these ratios, which are based on lifetimes established in separate fatiguing experiments, are taken to be linearly additive, one obtains the law of cumulative damage in its simplest form as proposed by Miner:[1]

$$\frac{t_1}{(t_b)_1} + \frac{t_2}{(t_b)_2} + \frac{t_3}{(t_b)_3} + \dots \frac{t_f}{(t_b)_f} = \sum_{i=1}^{f} \frac{t_i}{(t_b)_i} = 1 \tag{1}$$

and

$$(t_b)_{\text{est}} = \sum_{i=1}^{f} t_i$$

where t_i and $(t_b)_i$ are the time the specimen is held under constant stress σ_i and the time to failure under σ_i, respectively, and $(t_b)_{\text{est}}$ is the total time to rupture for the sequence of applied stresses σ_i ($i = 1, 2, 3, \dots, f$).

For an experiment under periodic loading this law may be written

$$\frac{n_1}{N_1} + \frac{n_2}{N_2} + \frac{n_3}{N_3} + \dots \frac{n_f}{N_f} = \sum_{i=1}^{f} \frac{n_i}{N_i} = 1 \tag{2}$$

where n_i is the number of cycles to which a sample is subjected short of failure and N_i is the average number of cycles to failure at a particular level of cyclic stress, represented by the subscript i.

On being applied to an experiment where the stress is changing continuously with time, e.g., in a tensile test, $\sigma = \sigma$ (t), the linear law of cumulative damage assumes the form

$$\int_0^{t^*} \frac{dt}{t_b[\sigma(t)]} = 1 \tag{3}$$

Here t^* is the total time to rupture under the selected form of $\sigma(t)$.

Several attempts to establish experimentally the applicability of this law were reported. While some authors found that their data conformed satisfactorily to the predicted lifetimes, there were also cases where the deviations were appreciable. So for example, Corten and Dolan[2] proposed on the basis of their results that the cumulative damage expression should be modified as follows:

$$t_1/(t_b)_1 + \sum_{i=2}^{f} t_i/(t_b)_i \, (\sigma_i/\sigma_1)^a = 1 \tag{4}$$

where a is a material constant. Still another form of cumulative damage hypothesis was put forward by Williams.[3] This author suggested the following form

$$\sum_{i=1}^{f} (t_i/(t_b)_i)^a = 1 \tag{4a}$$

for experiments under various strain rates ϵ_i.

Freudenthal and Heller[4] made an extensive study of fatigue under the loading conditions encountered in aircraft structures. They argue that the introduction of nonlinear functions of damage accumulation does not result in more reliable performance estimates. They recognize that even a very short intermittent application of a higher stress level σ_{i+m} should result in an accelerated damage accumulation in relation to that associated with constant stress level σ_i. Therefore, they introduce the concept of stress interaction factors $W_i > 1$ and modify the accumulative damage rule as follows:

$$\sum_{i=1}^{f} (n_i W_i/N_i) = 1 \tag{4b}$$

Values of individual stress interaction factors which increase with decreasing load must be determined experimentally. It should be noted, however, that use of eqs. (4), (4a), and (4b) requires the determination of the factors a or W_i which involves the performance of special and lengthy cumulative damage experiments. These experiments being very seldom performed, the above equations are of very limited use. Since the scope of this work is to explore the possibilities of predicting the performance in end-use applications from data obtained in standard testing, our discussions will be limited to the use of eqs. (1–3) only.

The major problem with experimental approaches aimed at a verification of a proposed model of material endurance in fatigue is the inconsistent scatter in measured lifetimes. This scatter is observed despite the fact that the tested specimens are selected to be as nearly identical as possible. Thus, the time to failure $(t_b)_i$ is actually an average lifetime determined on the basis of several repeated experiments and, in addition, some of the specimens may fail before the last level of stress which introduces further uncertainty in the calculations. The experimental results had therefore to be considered with caution and, in our opinion, none of the reported data on this subject satisfactorily answered the basic questions: what is the accuracy of lifetimes predicted by eqs. (1–3), does the accuracy depend on material properties, and how is the accuracy affected by changes in experimental conditions?

Despite the fact that the experimental data often agree reasonably with the estimates obtained by eqs. (1)–(3), the hypothesis on which these equations are based remains open to criticism. Alfrey[5] pointed out in his discussion of this problem that the time to failure t_b depends on two stress-dependent quantities; the rate of internal breakdown $B'(\sigma)$ and critical damage at break $B_c(\sigma)$ as follows:

$$t_b(\sigma) = B_c(\sigma)/B'(\sigma) \tag{5}$$

Thus, for an experiment where σ_1 is applied for time t_1 and then stress σ_2 is applied until the specimen breaks the following expression should apply:

$$t_1(\sigma_1)B'(\sigma_1) + t_2(\sigma_2)B'(\sigma_2) = B_c(\sigma_2) \tag{6}$$

If, however, stress σ_1 is applied until the specimen ruptures, then

$$t_1(\sigma_1)B'(\sigma_1) + [t_b(\sigma_1) - t_1(\sigma_1)]B'(\sigma_1) = B_c(\sigma_1) \tag{7}$$

applies, where $t_b(\sigma_1)$ is the time to failure under uninterrupted stress σ_1.

In order that eqs. (6) and (7) be consistent with eq. (1)

$$\frac{t_1(\sigma_1)B'(\sigma_1)}{B_c(\sigma_2)} \cong \frac{t_1(\sigma_1)B'(\sigma_1)}{B_c(\sigma_1)} \tag{8}$$

or

$$B_c(\sigma_2) \cong B_c(\sigma_1) \tag{9}$$

Thus, the necessary condition for the applicability of eq. (1) is that B_c is affected little by changes in stress. In addition, one would expect that the rate of breakdown B' may not be a function of stress only but would depend also on the degree of breakdown already present. This argument further increases doubts that expressions as simple as eqs. (1)–(3) could give satisfactory fits to experimental data obtained in cumulative damage experiment.

THEORY AND CALCULATIONS

In materials that break in a brittlelike fracture, the breakdown is localized; the undamaged sections of ruptured specimens show relatively less important effects of straining. The breakdown is initiated from a pre-existing defect which under applied stress develops into a macrocrack. Below a critical size this macrocrack is stress-sustaining and grows slowly, presumably by a thermally activated process. At a size slightly larger than critical, the crack becomes unstable and propagates spontaneously across the specimen to produce rupture.[6]

For systems where cracks are circular, flat, and perpendicular to the applied stress, the free energy Δf_r associated with their formation, according to Sack,[7] is given by

$$\Delta f_r = 2\pi r(\rho + p) - 8(1 - \mu^2)r^3\sigma^2q^2/3E \tag{10}$$

where r is the radius of the crack, ρ is the specific surface energy defined as the work required to form a unit area of surface in a brittle material by the formation of planes each one-half unit in area, p is the work of plastic deformation, E is Young's modulus, μ is Poisson's ratio, and q is the stress concentration factor associated with the pre-existing defect. The curve of free energy Δf_r versus r has a maximum at $r^* = (\rho + p)E/2(1 - \mu^2)q^2\sigma^2$. Thus, as soon as this radius is exceeded, the specimen fractures catastrophically. The area $\pi(r^{*2} - r_0^2)$ where r_0 is the radius of the pre-existing crack, should then be considered equivalent to the critical damage at break B_c discussed above. This area is, as Alfrey[5] pointed out, stress-dependent and decreases with increasing stress.

Thus, in a series of experiments where the stress σ_1 is applied first for a time t_1 and then the specimens are stressed to rupture, either under a larger stress σ_2 in time t_2 or under a smaller stress σ_3 in time t_3, the damage ratio expressed as the ratio of crack growth produced by the stress σ_1 in time t_1 [damage (σ_1, t_1)] to the critical radius of the crack at break [damage crit. (σ_1)] is smaller for the experiment ending at a higher stress σ_3.

In general

$$\frac{\text{damage } (\sigma_1,t_1)}{\text{damage crit. } (\sigma_2,t_2)} \lessgtr \frac{\text{damage } (\sigma_1,t_1)}{\text{damage crit. } (\sigma_3,t_3)} \tag{11}$$

depending on whether σ_2 is less or greater than σ_3.

For systems where the rate of crack growth is relatively independent of previous stress history, damage ratios would equal approximately expended lifetime ratios. As a consequence of the above considerations one would expect that for experiments where the stress increases with time, the failure would occur at values of the sum $\Sigma t_i/(t_b)_i$ less than unity, while in experiments where the stress decreases with time $\Sigma t_i/(t_b)_i$ should exceed unity at the time of rupture.

If, however, the rate of crack growth depends on the stress history, damage ratios no longer equal expended lifetime ratios and the errors associated with the application of eqs. (1)–(3) could not be estimated unless the expression for the rate of crack growth were known. The growth of cracks and rupture of fibrous polymers have been recently discussed by Prevorsek and Lyons.[6,8] Assuming that the rupture is a result of the growth of a circular crack perpendicular to the fiber and to the applied stress, these authors arrived at the following expression for the rate of polymer chain breakdown at the tip of a crack.

$$R(\sigma,r) = \frac{2\pi rkTZ}{hl} \exp\left\{-\frac{\Delta f_r}{kT} - \frac{\Delta F^*}{kT} + \frac{vq^2\sigma^2}{2EkT}\right\} \tag{12}$$

Here Z is the concentration of nucleation sites, l is the average distance between polymer chains, k is the Boltzmann constant, and h is the Planck constant.

If, as experimental results appear to indicate, the rupture of a specimen is principally the process of initiating an unstable crack, the growth areas required to cause the instability should slightly exceed $\pi(r^{*2} - r_0^2)$.

Consequently, the time to failure could then be approximated by the time required to increase the radius of a crack from r_0 to $r^* + \delta r$, δr being a small increment accounting for the fact that the crack having radius r^* is still stress-sustaining and that there is a finite time associated with the propagation of the unstable crack across the specimen. In the calculations presented below it was assumed that $\delta r = l/20$. It is plausible that this assumption introduces an error which varies with experimental conditions since δr itself may be a function of σ, ρ, E, and ΔF^*. However, the time required for the crack to grow from r^* to $r^* + \delta r$ should in general be small compared with the time corresponding to the growth from r_0 to r^*. Thus, the error attendant on the assumption that δr invariably equals $l/20$ should have little effect on the final results.

With these approximations the time to failure under constant stress assumes the form

$$t_b(\sigma) = K[(r^* + \delta r)^2 - r_0^2]\pi/R(r,\sigma)A_m \tag{13}$$

where $2A_m$ is an average surface area formed when two polymer chain segments are separated by the moving tip of the crack, and K is a coefficient which takes into account that the fracture is localized and that internal breakdown takes place primarily in areas of high concentration of stress. Considering that the rate $R(r\cdot\sigma)$ given by eq. (12) is expressed as the number of chain segments participating in crack growth throughout the unit volume of the specimen in unit time, K can be interpreted as the ratio of the breakdown occurring in the specimen under consideration, to the breakdown which could occur in an ideal uniform specimen of unit volume.

No general method exists for finding K. However, it is reasonable to assume that for a given specimen its value is affected little by changes in the applied stress. Thus, for the purposes of this work K can be given an arbitrary value.

Since the rate of crack growth $R(r,\sigma)$ is a function of the radius of the growing crack r, the determination of the time to break $t_b(\sigma)$ by using eq. (13) has been made by means of a set of equations of the type*

$$t_i = (2\pi K r_i/l)/R(r_i,\sigma) \tag{14}$$

with r_i assuming discrete values between r_0 and $r^* + \delta r$ according to

$$r_i = r_0 + k\Delta r \qquad k = 0, 1, 2, 3, \ldots, k_m$$

Here Δr is an arbitrary small increment in r and k_m is the lowest value of k satisfying the condition

$$[r_0 + (k_m + 1)\Delta r] \geq r^* + \delta r \tag{15}$$

That is, the rate $R(r_i,\sigma)$ was maintained constant until the area produced by breakage of molecules under this condition equaled that associated with an increase in crack radius from r_i to r_{i+1}. Then, $R(r_i,\sigma)$ was changed stepwise to the rate $R(r_{i+1},\sigma)$. From each equation, the time t_i was calculated and the procedure continued until $k = k_m$.

The calculation of the duration of this last step which terminates with the rupture of the specimen has to account for the fact that $r^* + \delta r$ which is a function of E, ρ, ΔF^* and σ does not necessarily equal $r_0 + k_m\Delta r$ but falls within the bounds

$$r_0 + k_m\Delta r < r^* + \delta r \leq r_0 + (k_m + 1)\Delta r$$

Consequently, the bounds for the number of molecules required to break in the last step M_f in order to produce the rupture are given

$$0 < M_f \leq [2\pi(r_0 + k_m\Delta r)/l] \tag{16}$$

If we write that

$$M_f = m[2\pi(r_0 + k_m\Delta r)/l] \tag{17}$$

then m can be regarded as the fraction of molecules in contact with the crack having radius $r_0 + k_m\Delta r$ that must be broken in order to increase its size by an area equal to that formed if the radius of the crack increases from $r_0 + k_m\Delta r$ to $r^* + \delta r$. We estimated m using

$$r^* + \delta r = r_0 + (k_m + m)\Delta r \tag{18}$$

which leads to

$$m = [(r^* - r_0 + \delta r)/\Delta r] - k_m \tag{19}$$

* Strictly, eq. (14) implies that polymer chains are perfectly oriented and located on concentric equidistant circles. Since changes in arrangements of molecules would have only a minor effect on the overall rate of crack growth, the conclusions based on the results of this calculation should not be restricted to this model, but have very probably general applicability.

Hence, the duration of the last step t_f takes the form

$$t_f = K \frac{2\pi(r_0 + k_m\Delta r)}{l}\left(\frac{r^* - r_0 + \delta r}{\Delta r} - k_m\right)\Big/R(r_0 + k\Delta r,\sigma) \quad (20)$$

and time to failure

$$t_b(\sigma) = \sum_{i=1} t_i \quad (21)$$

The σ–log t_b relationship obtained by eq. (13) and the calculating procedure described above agree with the reported experimental data which indicate that log (times to failure) plotted as function of the applied stress usually falls close to a straight line. There is, however, a difference between the present theory of strength and those proposed previously; while the theories of Bueche[9,10] and Coleman and Knox[11,12] predict a linear σ–log t_b relationship, the plots obtained from eq. (13) have a slight curvature towards the stress axis. The deviation from the linear relationship, however, is very small and may be difficult to verify experimentally. A recent theory by Tung[13] predicts a linear relationship between log σ and log t_b. The effects of various parameters on σ–log t_b curves obtained by eq. (13) is shown in Appendix I.

It should be pointed out that in addition to differences due to the dissimilarities in the physical interpretation of the cause of failure assumed in the proposed models, there is another essential disparity between the expression for the time to break given by eq. (13) and those derived, e.g., by Bueche,[9,10] Coleman,[11,12] and Tung.[13] Namely, t_b in eq. (13) does not depend only on material parameters, stress and temperature, but is also a function of the radius of the crack which finally causes the rupture. Since this latter quantity is a function of stress history, eq. (13) can be used to predict lifetimes under random loading conditions, whereas expressions derived by the above-mentioned authors are not applicable to treat such experimental conditions.

The calculating procedure to obtain $t_b[\sigma(t)]$, where $\sigma(t)$ is a rectangular step function with σ_k being constant during time intervals t_k at the end of which stress suddenly assumes another value σ_{i+1}, was analogous to that described above for $t_b(\sigma)$, with the exception that at time Σt_k ($k = 1, 2, 3, \ldots,$) ending the intervals t_k the rate $R(r_i,t_k)$ was changed to the rate $R(r_i,t_{k+1})$.

RESULTS AND DISCUSSION

The applicability of eq. (1) for predicting lifetimes in cumulative damage testing was examined in a series of simulated experiments all involving eight different levels of stress. Values of these stress σ_k were so selected that the corresponding times to failure under constant stress σ_k fell in the range of 10^{-2} to 10^6 sec., while time intervals t_k of application of these stresses in each simulated experiment were such that the failure predicted by eq.

(1) occurred at time $t = \sum_{k=1}^{8} t_k$. In the discussion of data and presented tables the indices of stresses k ($k = 1, 2, 3, \ldots 8$) refer to the magnitude of σ and increase with increasing stress, while the indices j ($j = 1, 2, 3, \ldots, 8$) refer to the order of application of stress.

For each programed experiment, the times to failure $t_b[\sigma(t_k)]$ were calculated from eq. (13) by using the calculating procedure described above and were compared with the lifetime estimates $(t_b)_{est}$ obtained from eq. (1). Times to failure $(t_b)_k$ under constant stress k needed for the calculations of expended lifetimes $t_k/(t_b)_k$ appearing in eq. (1) were also calculated by using eq. (13).

The discrepancies between these two estimates were then expressed as the ratio $D = t_b[\sigma(t_k)]/(t_b)_{est}$. Since the model on which the eq. (13) was derived is physically much more realistic than that leading to eq. (1) we assumed that in cases where the values of this ratio do not vary appreciably from unity, the use of eq. (1) to estimate lifetimes may be justified. If, however, the difference is large, the usefulness of eq. (1) is questionable. Values of the ratio D higher (lower) than unity indicate that the rupture would probably occur in a longer (shorter) time than that estimated by eq. (1).

The objective of these calculations was to determine for which cases eq. (1) may give useful estimates for expected lifetimes and more specifically, to establish the effect on the value of the ratio D resulting from changes in: (*a*) sequences of applied stresses; (*b*) durations of individual periods of constant load application; (*c*) values of structural parameters ρ, E, and ΔF^*; and (*d*) overall magnitude of applied stresses.

Change in Stress Sequence

Five experiments were programed, each involving the same eight stresses σ_k. Indexing these stresses from 1 to 8 according to their increasing values, these sequences can be represented as follows: No. 1 [1,2,3,4,5,6,7,8]; No. 2 [8,7,6,5,4,3,2,1]; No. 3 [2,6,1,3,8,5,7,4]; No. 4 [5,8,6,3,1,4,2,7]; No. 5 [6,3,1,7,4,8,5,2], where the numerals in the brackets represent stress indices appearing in the same order as applied in the experiments. Unless stated otherwise, the times t_k of constant stress application were selected so that the corresponding expended lifetimes $t_k/(t_b)_k$ all equaled 1/8. Representative results for two combinations of structural parameters ρ, E, and ΔF^* appear in Tables II–V; values of other parameters used in the calculations are given in Table I. For the experiments where the stress increases with time, the lifetimes predicted by eq. (1) agree almost perfectly with those calculated by using eq. (13). Furthermore it is shown in Appendix II that the difference between $(t_b)_{est}$ and $t_b[\sigma(t)]$ is affected very little by changes in the structural parameters ρ, E, and ΔF^* or by changes in expended lifetimes. The ratio D is always smaller than unity but seldom smaller than 0.99. Equation (1) can therefore be regarded as the upper bound for the expected lifetime in experiments in which the stress increases

TABLE I
Numerical Values of Parameters Used in the Calculations

Parameter	Value
Temperature T, °K.	300°
Poisson's ratio μ	0.4
Volume of specimen V, cm.^{3}	3×10^{-5}
Concentration of nucleation sites Z, cm.$^{-3}$	2×10^{-23}
Average distance between polymer chains l, cm.	5×10^{-8}
Plastic deformation p, erg/cm.^{2}	0
Stress concentration factor q	10
Initial radius of crack r_0, cm.	10^{-10}

TABLE II
Values of Stresses σ_k and Corresponding $(t_b)_k$ as Calculated from Equation (13) and Used in Simulated Experiments Described in Table III, with $E = 10^{10}$ dyne/cm.^{2}, $\rho = 500$ erg/cm.^{2}, ΔF^* 3.5×10^{-12} erg/segment, and Other Parameters as in Table I

k	Stress $\sigma_k \times 10^{-8}$, dyne/cm.^{2}	Time to failure $(t_b)_k$, $\times 10^{-6}$, sec.
1	18	1.21715970
2	20	0.17648155
3	22	0.02088164
4	24	0.00213719
5	26	0.00017968
6	28	0.00001240
7	30	0.00000075
8	32	0.00000004

TABLE III
Times to Break Calculated by Equation (13) Compared to Those Estimated by Equation (1) for Five Sequences of Stresses, Using Parameters from Tables I and II, All Expended Lifetimes $t_k/(t_b)_k = {}^1/_8$

Stress sequence[a]	Time to break $(t_b)_{est} \times 10^{-6}$ from eq. (1), sec.	Time to break $t[\sigma(t_k)] \times 10^{-6}$ from eq. (13), sec.	$D = \frac{t_b[\sigma(t_k)]}{(t_b)_{est}}$
No. 1 (1,2,3,4,5,6,7,8)	0.17710662	0.17710659	0.9999998
No. 2 (8,7,6,5,4,3,2,1)	0.17710662	0.44681814	2.5228766
No. 3 (2,6,1,3,8,5,7,4)	0.17710662	0.17683947	0.9984916
No. 4 (5,8,6,3,1,4,2,7)	0.17710662	0.17710652	0.9999994
No. 5 (6,3,1,7,4,8,5,2)	0.17710662	0.20247584	1.1432426

[a] Stress indices k appear in the same order as applied in the experiment; values of k increase with increasing stress.

with time. Then $(t_b)_{est} \approx t_b[\sigma(t_j)]$ ($j = 1,2,3, \ldots,$) provided $\sigma_1, < \sigma_2 < \sigma_3$ (where j refers to the order of the applied stresses, σ_1 is the first stress applied in the experiment, σ_2 the second, σ_3 the third, etc.

In cases where stress decreases with time, lifetimes predicted from eq. (1) are consistently shorter than those obtained from eq.(13). The differences are often very large and may exceed several hundred per cent for experi-

TABLE IV

Values of Stresses σ_k and Corresponding $(t_b)_k$ as Calculated from Equation (13) and Used in Simulated Experiments Described in Table VI, with $E = 10^{11}$ dyne/cm.^{2}, $\rho = 500$ erg/cm.^{2}, $\Delta F^* = 3.5 \times 10^{-12}$ erg/segment, and Other Parameters as in Table I

k	Stress $\sigma_k \times 10^{-8}$, dyne/cm.	Time to failure $(t_b)_k \times 10^{-6}$, sec.
1	60	0.48696395
2	65	0.09736286
3	70	0.01806439
4	75	0.00296115
5	80	0.00042869
6	85	0.00005844
7	90	0.00000662
8	95	0.00000071

TABLE V

Times to Break Calculated by Equation (13) Compared to Those Estimated by Equation (1) for Five Sequences of Stresses, Using Parameters from Tables I and IV, All Expended Lifetimes $t_k/(t_b)_k = {}^1/_8$.

Stress sequence[a]	Time to break $(t_b)_{est} \times 10^{-5}$, from eq. (1), sec.	Time to break $t[\sigma(t_k)] \times 10^{-5}$, from eq. (13), sec.	$D = \frac{t_b[\sigma(t_k)]}{(t_b)_{est}}$
No. 1 (1,2,3,4,5,6,7,8)	0.75730853	0.75730654	0.9999974
No. 2 (8,7,6,5,4,3,2,1)	0.75730853	1.65587280	2.1865234
No. 3 (2,6,1,3,8,5,7,4)	0.75730853	0.75360710	0.9951124
No. 4 (5,8,6,3,1,4,2,7)	0.75730853	0.75730026	0.9999891
No. 5 (6,3,1,7,4,8,5,2)	0.75730853	0.85339027	1.1268726

[a] Stress indices k appearing in the same order as applied in the experiment; values of k increase with increasing stress.

ments where expended lifetimes associated with various stresses are of the same magnitude. Thus, for experiments where $\sigma_1 > \sigma_2 > \sigma_3 \ldots (t_b)_{est} < t_b[\sigma(t_j)](j = 1, 2, 3, \ldots,)$, where j refers to the order of the applied stress.

Equation (1) represents in this case the lower bound for expected lifetimes. It should be noted, however, that the accuracy of the estimate appears to be in this latter case much lower than that found with a sequence of stresses increasing with time. Stress sequence [2,6,1,3,8,5,7,4,1] (No. 3) and [5,8,6,3,1,4,2,7] (No. 4) give results similar to those obtained with increasing sequence No. 1, while the sequence [6,3,1,7,4,3,5,2] (No. 5) behaves more like [8.7,6,5,4,3,2,1]. Thus, with sequences No. 3 and 4, eq. (1) gives lifetimes which are consistently shorter than those obtained from eq. (13), but the values of the ratio D are usually very close to unity. With sequence No. 5 the error is much larger, and values of D usually exceed unity. It should be pointed out, however, that it is also possible to select the experimental variables (t_k, E, ρ, and ΔF^*) in such a way as to obtain values of $D < 1$. The discussion of this somewhat unexpected effect is presented in the appendix.

The significance of these results becomes more apparent if consideration is given to the number of steps involving an increase or a decrease in stress, that is, if the sequences are regarded as predominantly increasing or decreasing in stress. With sequences No. 3 and 4 involving 4 (3) and 3 (4) increasing (decreasing) steps in stress, respectively, and which therefore can not be regarded as either predominantly decreasing or increasing, eq. (1) seem to give reasonable lifetime estimates. With the predominantly decreasing sequence No. 5 which involves 5 (2) decreasing (increasing) steps in stress, D usually exceeds unity, as found previously with the decreasing sequence No. 2.

For experiments in which stresses neither decrease nor increase consistently with time, eq. (1) may thus give either larger or lower estimates than eq. (13), the sign of the differences being dependent on the number and magnitude of decreasing or increasing steps.

It was further shown that with the decreasing stress sequences, No. 2 and 5, the changes in the duration of constant load applications t_k and the changes in values of the structural parameters ρ, E, and ΔF^* can affect the ratio D significantly. However, the study of these effects, although interesting and important from an engineering point of view, did not alter in any way the general conclusions put forward in this chapter. Thus, their discussion is presented in the appendix.

Comparison With Experiments

One of the objectives of this study is to compare the predictions of the calculations discussed above with the results obtained when the lifetime data determined in cumulative damage experiments are analyzed by means of eqs. (1)–(3). The scope of such a comparison is to confirm the physical reality of the model on which eq. (13) has been derived and to provide experimental verification for the conclusions put forward in this paper. The establishing of σ–log t_b, or σ–log N relationships which are required for the calculation of expended lifetimes, involves numerous time-consuming experiments. In addition, these relationships have to be known with great accuracy to ascertain meaningful results. Thus, the discussion is limited to the previously reported data on the subject.

It is interesting that σ–log t_b relationships obtained from eq. (13) are similar to those predicted by theories developed by Bueche[9,10] and Coleman[11,12] which assume different breaking mechanisms (see Appendix I) and agree also with the data obtained with nonpolymeric materials such as metals. Thus, one can expect that the qualitative conclusions put forward in this paper may not be restricted to the type of fracture assumed in the derivation of eq. (13) but may have general applicability. The discussion of experimental data will therefore not be limited to experiments on fibers but will include also results obtained with metals with which the law of cumulative damage has been most extensively studied.

It should also be pointed out that eqs. (1)–(3) represent the same hypothesis applied to various types of load histories. Thus, the qualitative considerations regarding the applicability of this hypothesis to one type of experiment (e.g., fatigue in cyclic loading) can be generalized and applied to experiments conducted under different load histories (e.g., static loading), provided that the changes in experimental conditions do not affect appreciably the mechanism of rupture.

The law of cumulative damage as expressed in eqs. (1)–(3) is an empirical law. The fact that the average value of the sum of cycle ratios at failure $(\Sigma n_i/N_i)_{\text{avg}}$, calculated for the results of 21 experiments involving two or more levels of cyclic loading, equaled 1.015, led Miner[1] to assume that the failure occurs when $\Sigma n_i/N_i = 1$. No attention was given to the fact that maximum and minimum values were 1.49 and 0.80, and no attempts were made to find out whether values of $\Sigma n_i/N_i$ higher or lower than unity are found preferentially with a particular type of experiment.

In view of the results of our calculation, it is interesting to compare the data obtained in experiments involving decreasing stress with those found in experiments where stress increases with time. Five experiments of the former type were performed involving two or more stresses, the results were as follows: 1.45, 0.98, 1.03, 0.99, and 1.32, which gives an average of 1.15. With experiments involving increasingly severe conditions values of $\Sigma n_i/N_i$ were lower: 0.80, 1.00, 0.75, and 1.11, which gives an average of 0.91.

In order further to illustrate the effect of changes in the stress sequence, Table VI gives values of expended lifetimes n_i/N_i and their sum at failure $\Sigma n_i/N_i$ obtained in two experiments, each involving three similar levels of stress; in one experiment the stress increased, in the other it decreased with time.

TABLE VI
Results of Cumulative Damage Experiments Involving Three Stress Levels[a]

Maximum stress, psi	n_i, cycles	N_i, cycles	$\frac{n_i}{N_i}$	$\sum \frac{n_i}{N_i}$
$\sigma_1 = 4300$	7.2	31.5	0.44	
$\sigma_2 = 3515$	15.6	64.0	0.24	1.45
$\sigma_3 = 2735$	120.0	155.0	0.74	
$\sigma_1 = 2735$	72.0	155.0	0.46	
$\sigma_2 = 3615$	20.0	64.0	0.31	0.80
$\sigma_3 = 3900$	1.2	45.0	0.03	

[a] Data of Miner.[1]

Thus, it appears that Miner's experimental data agree at least qualitatively with the calculations presented in this paper. The fact that some values of $\Sigma n_i/N_i$ such as 0.98 and 0.99 found in decreasing sequences or 1.11 measured in an increasing sequence do not follow the expected trend could be easily explained when scatter in the lifetimes and difficulties in obtaining accurate σ–log N relationships are taken into account. Therefore, values of 0.98 and 0.99 can not be regarded in this case as being significantly lower than unity and among the reported data there is only one value of $\Sigma n_i/N_i$ (1.11 with increasing sequence) that could be considered contradictory to our calculations.

A more elaborate investigation of the applicability of eq. (1) has been recently reported by Lyons and Prevorsek.[14] Experiments that were carried out on fibers fatigued in cyclic longitudinal tension involved six increasingly severe conditions, altogether 175 specimens being ruptured. The analysis of data indicated that on the average the specimens failed at a value of $\Sigma n_i/N_i = 0.968$. Because of the large number of specimens tested the result was believed to be significant, and support to the assumption that for this type of experiment (stress increasing with time) eq. (1) may be useful to predict the lifetimes.

These latter data then substantiate both conclusions put forward above regarding the applicability of eqs. (1)–(3) to experiments in which stress is made increasingly severe with time, namely, that: (*a*) the lifetimes predicted by eqs. (1)–(3) should be longer than the measured ones, and (*b*) that the discrepancies between measured and predicted values should be small.

The reported experimental data, while useful for qualitative considerations, are insufficient for quantitative comparisons between predicted and observed values of the ratio D. A quantitative analysis would be possible only if the material parameters ρ, E, and ΔF^* were known for the materials used in fatiguing experiments.

CONCLUSIONS

The presented theory can not be given at present a rigorous testing, and a quantitative verification will have to wait until more experimental data using carefully characterized specimens are available. The qualitative conclusions put forward, however, are in agreement with experiment. Consequently, the following points, useful to design engineers, and applicable to various materials can be advanced at this time.

(*1*) The performance of specimens in cumulative damage experiments which involve both decreasing and increasing steps in loading conditions can not be predicted from eqs. (1)–(3), since the size and magnitude of error are affected by stress pattern, material properties, duration of individual steps, etc. It is particularly important that in some cases variations in experimental conditions, such as changing the overall level of stressing may reverse the sign of error, which renders the estimates obtained from eqs. (1)–(3) completely unreliable.

(*2*) For experiments where loading conditions become decreasingly severe with time, the estimates obtained from eqs. (1)–(3) can be regarded as the lower bound. That means the specimens will on the average survive to longer times than predicted, thus, the use of eqs. (1)–(3) for design purposes can not lead to unsafe structures.

(*3*) For experiments where the loading conditions become increasingly severe with time, the estimates obtained from eqs. (1)–(3) can be regarded as the upper bound, and the specimens will on the average fail sooner than estimated.

(*4*) The calculations further indicate that for experiments where loading conditions become increasingly severe with time the difference between predicted and observed lifetimes should be very small. This, probably the most interesting conclusion of the presented calculations, is, however, not sufficiently supported by experiment. While the data reported by Lyons and Prevorsek[14] agree with this point, the results obtained by Miner[1] seem to indicate that the error might be larger than that predicted by our calculations. Thus, at the present time, the reliability of eqs. (1)–(3) for this specific case must be regarded as uncertain. It should also be emphasized that in this point we no longer consider a qualitative prediction based on our calculations but a quantitative one. Consequently, no generalization could be made with regard to this conclusion even if further supporting data were obtained, unless it were established that the accuracy of the predictions based on $\Sigma n_i/N_i = 1$ is affected little by changes in rupture mechanisms. In spite of this limitation, the experimental verification of this latter conclusion for systems of interest would be of considerable value, since fatiguing experiments involving increasingly severe conditions represent an important and frequently used type of material evaluation (e.g., accelerated testing). Were the results affirmative, then the analysis of such experimental data would be greatly facilitated through use of eqs. (1)–(3).

APPENDIX I

σ–t_b Relationship Obtained by Equation (13)

Comparison of theoretical and experimental stress versus time-to-break relationships can serve several purposes; it can be used to verify the model on which a theory is based, it provides information concerning possible rupture mechanisms, or it can be used to extend the experimental data in the region of experimental conditions not attainable by existing experimental set-ups, etc. Unfortunately, it is difficult and time consuming to obtain reliable σ–t_b relationships over wide ranges of stresses because of the wide scatter in the measured lifetimes that cannot be avoided. Furthermore, changes in stress may have an effect on the rupture mechanism or may alter in some other way the material responses, which makes the analysis even more complex. In spite of these uncertainties, it is interesting to compare σ–t_b relationships predicted by various theories proposed for polymeric materials. The theory of Bueche[9,10] which assumes a critical stress predicts a linear σ–log t_b, relationship and so does the theory of Coleman and Knox[11,12] based on the concept of critical strain. Tung's[13] critical strain theory, however, predicts a linear relationship between log σ and

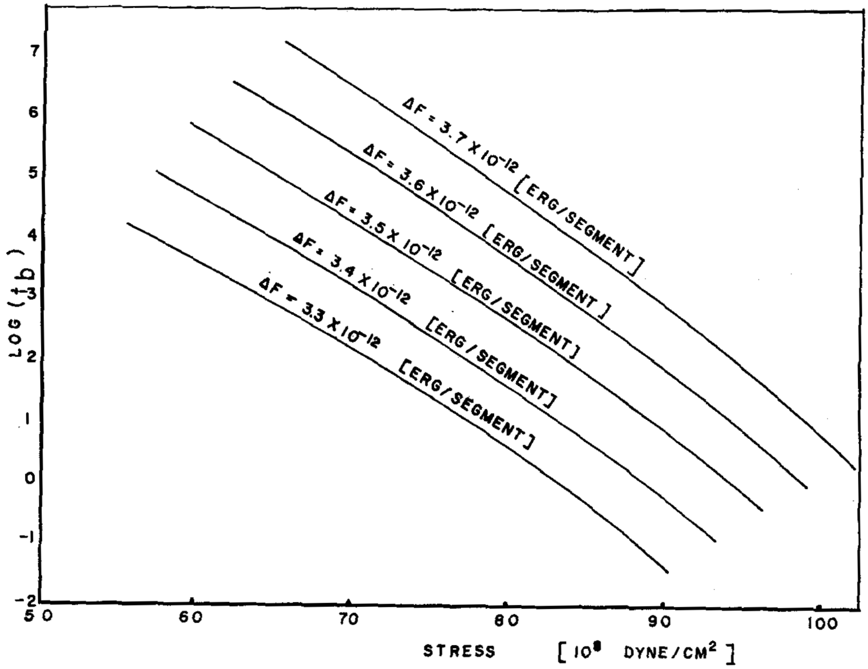

Fig. 1. σ–log t_b relationships calculated by eq. (13) for various ΔF^*, $Q = 500$ erg/cm.^{2}, $E = 10^{11}$ dyne/cm.^{2}; values of other parameters as in Table I.

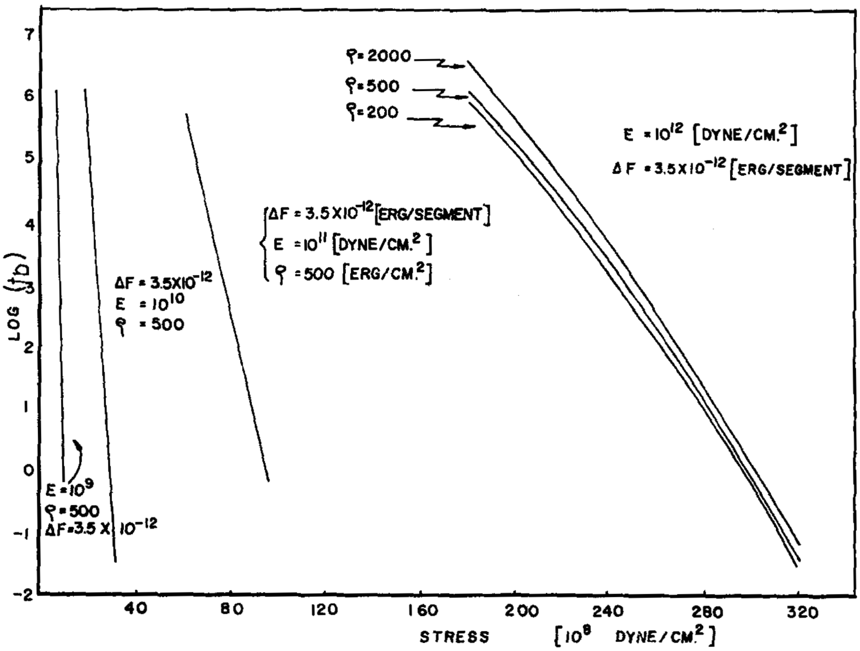

Fig. 2. σ–log t_b relationships calculated by eq. (13) for various E and ρ; $\Delta F^* = 3.5 \times 10^{-12}$ erg/segment, values of other parameters as in Table I.

log t_b. The critical crack theory of Prevorsek and Lyons[6,8] predicts a relationship which is almost linear when log t_b is plotted as a function of stress.

Stress–log time-to-break relationships obtained by eq. (13) have not yet been published. Since they are difficult to calculate without the use of a computer we show in Figures 1 and 2 families of σ–log t_b curves obtained with various combinations of structural parameters ρ, E, and ΔF^*.

APPENDIX II

Changes in Duration of Constant Load Application

Three types of experiments were programed to study the effect of changes in the duration of constant load applications t_j on the value of the ratio D. In type A experiments the times t_j of constant load application were so selected as to give values of expanded lifetimes $t_j/(t_b)_j$ associated with stress σ_j ($j = 1, 2, 3, \ldots, 8$) all equal to $^1/_8$. In type B experiments expended lifetimes decreased assuming values ($^7/_8$, $^7/_{8^2}$, $^7/_{8^3}$, $^7/_{8^4}$, $^7/_{8^5}$, $^7/_{8^6}$, $^7/_{8^7}$, $^7/_{8^7}$), while in type C experiments there values increased according to ($^7/_{8^7}$, $^7/_{8^7}$, $^7/_{8^6}$, $^7/_{8^5}$, $^7/_{8^4}$, $^7/_{8^3}$, $^7/_{8^2}$, $^7/_8$). It should be noted that here the index j refers to the order of application (not magnitude) of the applied stress. Thus $t_1(t_b)_1$, $t_2/(t_b)_2$, . . ., represent expended lifetimes associated with the first, second, . . ., stress applied in the experiments.

These results appear for a particular choice of ρ, E, and ΔF^* in Tables VII and VIII. With the increasing stress sequence No. 1 the described changes in expended lifetimes did not produce significant changes in values of lifetimes obtained from eq. (1), which agree closely with those obtained by eq. (13). With other sequences the effects are larger. When the stress sequence No. 3 is programed according to a type B experiment, values obtained from eq. (1) agree well with those calculated by using eq. (13), when the same sequence is programed in a type C, the error is much larger and may exceed 10%.

TABLE VII
Values of Stresses σ_k and Corresponding $(t_b)_k$ as Calculated from Equation (13) and Used in Simulated Experiments Described in Table VIII, with $E = 10^9$ dyne/cm.^{2} $\rho = 200$ erg/cm.^{2}, $\Delta F = 3.5 \times 10^{-12}$ erg/segment and Other Parameters as in Table I.

k	Stress $\sigma_k \times 10^{-8}$, dyne/cm.^{2}	Time to failure $(t_b)_k \times 10^{-6}$, sec.
1	6.0	0.34783139
2	6.5	0.07174105
3	7.0	0.01405008
4	7.5	0.00226441
5	8.0	0.00034831
6	8.5	0.00004748
7	9.0	0.00000529
8	9.5	0.00000057

TABLE VIII
Effects of Varying Expended Lifetimes

Sequence no.[a]	Experiment type[b]	Time to break $(t_b)_{est}$ from eq. (1), sec.[c]	Time to break $t_b[\sigma(t_k)]$ from eq. (13), sec.[c]	$D = \frac{t_b[\sigma(t_k)]}{(t_b)_{est}}$
1	A	0.54536073×10^5	0.54536014×10^5	0.9999989
1	B	0.31239518×10^6	0.30883628×10^6	0.9886077
1	C	0.45789885×10^1	0.45782468×10^6	0.9998380
2	A	0.54536073×10^5	0.95116399×10^5	1.7441006
2	B	0.31239518×10^6	0.31533835×10^6	1.0094213
2	C	0.45789885×10^1	0.69206816×10^5	1.5113996×10^4
3	A	0.54536073×10^5	0.54492527×10^5	0.9992015
3	B	0.31239518×10^6	0.31219915×10^5	0.9993725
3	C	0.45789885×10^1	0.40895573×10^1	0.8931137
4	A	0.54536073×10^5	0.54535522×10^5	0.9999899
5	A	0.54536073×10^5	0.58379343×10^5	1.0704721
5	B	0.31239518×10^6	0.30454456×10^6	0.9748696
5	C	0.45789885×10^1	0.10172853×10^5	0.2221638×10^4

[a] Sequence nos. 1–5 refer to respective stress sequences (1,2,3,4,5,6,7,8), (8,7,6,5,4,3,2, 1), (2,6,1,3,8,5,7,4), (5,8,6,3,1,4,2,7), (6,3,1,7,4,8,5,2), indices k in parenthesis appear in the same order as applied in the experiment, values of k increases with increasing stress, thus $\sigma_1 < \sigma_2 < \sigma_3 \ldots$

[b] Experiment types A, B, and C refer to the respective sequences of expended lifetimes $t_j/(t_b)_j$ (1/8, 1/8, 1/8,...1/8), (7/8, 7/8², 7/8³,...7/8⁷, 7/8⁷), (7/8⁷, 7/8⁷, 7/8⁶, 7/8⁵, 7/8², 7/8).

[c] Times to break calculated by eq. (13) compared to those estimated by eq. (1), by using parameters from Tables I and VII.

Sequences No. 2 and 5 with decreasing stress are more sensitive to such changes; in type A experiments eq. (1) gives values smaller than eq. (13); when the same stress sequences are programed in type B the error is greatly reduced, while with type C experiments the values of lifetimes estimated from eq. (1) are several orders of magnitude shorter than those calculated from eq. (13). Provided that the values of other parameters are the same, the differences between these two estimates are smaller with the sequence No. 5 than with the sequence No. 2.

Changes in Structural Parameters

The scope of these calculations was to find out whether the conclusions reached on the basis of experiments conducted on one material can be applied to some other material that ruptures in a similar way, and to establish the conditions under which such a generalization could be made. It should be noted, however, that changes in structural parameters when programed at constant values of preselected stress σ_k under condition that the expended lifetimes remain constant, affect the duration of constant load applications t_k and alter the range of lifetimes associated with the selected stress.

In an attempt to separate these effects the calculations were programed in two ways; in one series of experiments changes in structural parameters

ρ, E, and ΔF^* were performed at constant values of stresses σ_k while the t_k were adjusted to satisfy the conditions $t_k/(t_b)_k = {}^1/_8$. In the other series of experiments involving changes in ρ, E, and ΔF^*, times t_k were maintained constant and values σ_k were adjusted to satisfy the condition that all the expended lifetimes be ${}^1/_8$.

It is evident that these two calculating procedures simulate the experiments where (*a*) the same loading conditions are applied to various types of specimens and (*b*) loading conditions are adjusted to obtain a similar performance provided that Miner's law applies.

With stress sequences No. 1, 3, and 4, changes in ρ, E, and ΔF^* programed at σ_k = constant produce only small effects on the ratio D. Its value is always smaller than unity but usually larger than 0.99. Thus, one might expect that for experiments in which the stress increases with time, eqs. (1)–(3) should predict values of lifetimes which are very close to those determined by experiment, provided that the specimens fail in a brittlelike fracture as assumed in the derivation of eq. (13). Typical results of these calculations are summarized in Table IX.

TABLE IX

Effects of ρ, E, and ΔF^* at σ_k = constant on the Ratio $D = t_b[\sigma(t_k)]/(t_b)_{est}$ at Stress Range from 6×10^8 to 320×10^8 dyne/cm.^{2}; all Expended Lifetimes $t_k/(t_b)_k = {}^1/_8$, Values of Other Parameters as in Table I.

Stress sequence[a]	ρ, erg/cm.^{2}	E, dyne/cm.^{2}	$\Delta F^* \times 10^{12}$, erg/segment	$D = \dfrac{t_b[\sigma(t_k)]}{(t_b)_{est}}$
1 (1,2,3,4,5,6,7,8)	300	10^{10}	3.2	0.9998856
	1000	5×10^{11}	3.8	0.9999970
	2000	10^9	3.5	0.9999906
	800	10^{11}	3.5	0.9999939
	500	10^{12}	3.5	0.9999998
3 (2,6,1,3,8,5,7,4)	200	10^9	3.5	0.9992015
	1000	10^{10}	3.5	0.9986673
	800	10^{11}	3.5	0.9953169
	500	10^{12}	3.5	0.9984916
	2000	10^9	3.5	0.9956668
4 (5,8,6,3,1,4,2,7)	200	10^9	3.5	0.9999899
	1000	10^{10}	3.5	0.9999996
	800	10^{11}	3.5	0.9999898
	2000	10^{12}	3.5	0.9999996
	500	10^{12}	3.5	0.9999994

[a] Stress indices k appear in the same order as applied in the experiment, values of k increase with increasing stress.

With stress sequence No. 2, an increase in ρ at σ_k = const. results in an increase in the time to failure $t_b[\sigma(t_k)]$ and in the value of the ratio D. An example of this behavior is shown in Table X. A different response is observed with stress sequence No. 5. For experiments programed with σ_k = constant and values of $E = 10^{11}$ dyne/cm.^{2} and $\Delta F^* = 3.5 \times 10^{12}$

TABLE X

Effect of ρ at σ_k = Constant on the ratio $D = t_b[\sigma(t_k)]/(t_b)_{est}$; values of σ_k as in Table V, $E = 10^{11}$ dyne/cm.^{2}, $\Delta F^* = 3.5 \times 10^{-2}$ erg/segment, Other Parameters as in Table I, All Expended Lifetimes $t_k/(t_b)_k = 1/8$.

Stress sequence[a]	ρ, erg/cm.^{2}	D
No. 2 (8,7,6,5,4,3,2,1)	200	1.7441007
	500	2.1865243
	800	2.3648131
	2000	2.8517818
No. 5 (6,3,1,7,4,8,5,2)	200	1.0704721[b]
	500	1.1268726[b]
	800	1.1966975[b]
	1200	1.2283230[b]
	1600	1.2960667[b]
	2000	0.8412045[c]
	2200	0.8426493[c]
	2400	0.8433604[c]
	2800	0.8433708[c]

[a] Stress indices k appear in the same order as applied in the experiment, values of k increase with increasing stress.

[b] Break during application of σ_2.

[c] Break during application of σ_8.

erg/cm.^{2}, D exceeds unity if ρ assumes values between 200 and 1600 erg/cm.^{2}, $dD/d\rho > 0$ in this range. When ρ is increased to 2000 erg/cm.^{2}, a sudden jump in D is noticed which now becomes smaller than unity and is affected little by further increases in ρ (see Table X).

This somewhat unexpected stepwise change in D is apparently associated with the fact that a change in ρ may, under certain experimental conditions, shift the failure from a time interval t_k to the time interval t_{k+i} or t_{k-i} ($i = 1, 2, 3, \ldots,$) depending on whether the change in ρ is positive or negative. This shift is due to the fact that an increase (decrease) in ρ increases (decreases) the critical radius and decreases (increases) the rate of crack growth [eq. (12)]. Thus, for example, if one selects for a given set of parameters ρ, E, and ΔF a set of stresses σ_k in such a way as to produce the failure during the application of the last stress in the experiment and then the ρ is incremented and times $t_b[\sigma(t_k)]$ and $(t_b)_{est}$ are calculated, one finds that there is a value of ρ at which a further increment shifts the failure into the time interval associated with a preceding stress. In the case where this preceding stress is larger than the last one, this increment in ρ results in a decrease in internal damage to produce rupture which in turn leads to a stepwise decrease in $t_b[\sigma(t_k)]$. The fact that eq. (1) does not account for variations in the critical crack size as a function of stress results then in the observed stepwise change in D. An example of such behavior is presented in Table X.

It is interesting to note that this sudden change in D does not occur if the experiments are programed with t_k = constant and with adjustments of stress σ_k. For example, with the value of ρ = 2000 erg/cm.^{2},

TABLE XI

Values of Stresses σ_k Chosen to Yield Constant Times to Failure $(t_b)_k$ for Two Sets of Structural Parameters by Using Equation (13) and Applied in Simulated Experiments Described in Table XII, with Values of Other Parameters as in Table I

k	Case 1[a] $\sigma_k \times 10^{-8}$, dyne/cm.^{2}	Case 2[b] $\sigma_k \times 10^{-8}$, dyne/cm.^{2}	$(t_b)_k$, sec.
1	10	184.6	0.29×10^{6}
2	12	191.7	0.79×10^{5}
3	14	199.0	0.19×10^{5}
4	16	206.9	0.39×10^{4}
5	18	215.5	0.67×10^{3}
6	20	224.1	0.10×10^{3}
7	22	233.3	0.125×10^{2}
8	24	243.3	0.125×10^{1}

[a] $E = 10^{10}$ dyne/cm.^{2}; $\Delta F^* = 3.2 \times 10^{-2}$ erg/segment; $\rho = 300$ erg/cm.^{2}.
[b] $E = 5 \times 10^{11}$ dyne/cm.^{2}; $\Delta F^* = 3.8 \times 10^{-12}$ erg/segment; $\rho = 100$ erg/cm.^{2}.

D was found to be 1.205 when the stresses were adjusted to give t_k values comparable to those used with $\rho = 200$ erg/cm.^{2}

In order to explore whether the effect of changes in ρ, E, and ΔF^* on the value of D might be reduced if the experiments were conducted at t_k = constant instead of σ_k = constant, we programed stress sequences No. 1, 2, and 5 using two sets of parameters ρ, E, and ΔF^*. Values of stresses σ_k chosen to yield constant times to failure $(t_b)_k$ for the two sets of structural parameters are given in Table XI; the corresponding values of the ratio D are shown in Table XII. It can be seen in Table XII that the changes in structural parameters affect considerably both the magnitude and the sign of error with sequences No. 2 and 5, while no significant effect was produced by these changes with sequence No. 1.

On the basis of these data one can infer that with the exception of stress histories involving increasingly severe loading conditions, the quantitative conclusions regarding the accuracy of predictions obtained by eqs. (1)–(3) cannot be generalized. The presented examples also further illustrate the hazard of applying Miner's law to experiments involving both decreasing

TABLE XII

Effect of Changes in ρ, E, and ΔF^* at t_k = constant on the Ratio $D = t_b[\sigma(t_k)]/(t_b)_{est}$; Values of Stresses as in Table XI, All Expended Lifetimes $t_k/(t_b)_k = 1/_8$.

Stress sequence[a]	$E = 10^{10}$ dyne/cm.^{2}, $\Delta F^* = 3.2 \times 10^{-13}$ erg/segment, $\rho = 300$ erg/cm.^{2}	$E = 5 \times 10^{11}$ dyne/cm.^{2}, $\Delta F^* = 3.8 \times 10^{-12}$ erg/segment, $\rho = 1000$ erg/cm.^{2}
No. 1 (1,2,3,4,5,6,7,8)	0.9998856	0.9999970
No. 2 (8,7,6,5,4,3,2,1)	3.2831606	1.5613931
No. 5 (6,3,1,7,4,8,5,2)	0.7972310	1.1544284

[a] Stress indices k appear in the same order as applied in the experiment, values of k increase with increasing stress.

and increasing steps in loading conditions. In these cases, the magnitude and sign of error is not affected only by the number and magnitude of increasing and decreasing steps in stress but depends also on the values of the structural parameters and on the overall level of stress.

One of the authors (D. P.) wishes to acknowledge that this work and that published in ref. 14 was prompted by the reviewer who criticized the use of $\int_0^{t^*} dt/t_b[\sigma(t)] = 1$ in the derivation of the expression for strength in ref. 8. The authors are indebted to the Allied Chemical Corporation for making their computing facilities available for this work.

References

1. M. A. Miner, *J. Appl. Mech.*, **A12,** 159 (1945).
2. H. T. Corten and T. J. Dolan, paper presented at International Conference on Fatigue of Metals, Institution of Mechanical Engineers and American Society of Mechanical Engineers, 1956.
3. M. L. Williams, in *Solid Propellant Rocket Research Progress in Astronautics and Rocketry,* M. Summerfield, Ed., Vol. I, Academic Press, New York, 1960.
4. A. M. Freudenthal and R. A. Heller, *J. Aero/Space Sci.*, **26,** 431 (1959).
5. T. Alfrey, Jr., *Mechanical Behavior of High Polymers,* Interscience, New York, 1948.
6. D. Prevorsek and W. J. Lyons, *J. Appl. Phys.*, **35,** 3152 (1964).
7. R. A. Sack, *Proc. Phys. Soc. (London),* **58,** 729 (1946).
8. D. C. Prevorsek, *J. Polymer Sci. A-2,* **4,** 63 (1966).
9. F. Bueche, *J. Appl. Phys.*, **28,** 784 (1957).
10. F. Bueche, *J. Appl. Phys.*, **29,** 1231 (1958).
11. B. D. Coleman, *J. Polymer Sci.*, **20,** 447 (1956).
12. B. D. Coleman and A. G. Knox, *Textile Res. J.*, **27,** 393 (1957).
13. L. H. Tung, *J. Polymer Sci. A,* **3,** 1045 (1965).
14. W. J. Lyons and D. C. Prevorsek, *Textile Res. J.*, **35,** 1048 (1965).

Résumé

L'applicabilité de la loi de Miner pour le dommage cumulatif en vue de prédire les durées de vie dans des expériences comportant des charges complexes, a été examinée. Les durées de vie estimées par $\Sigma(t_i/(t_b)i) = 1$ sont comparées avec celles calculées par l'expression pour le temps de rupture dérivé par Prevorsek et Lyon en admettant que le temps de rupture peût être approché par le temps de formation pour former une brisure instable. Pour les expériences dans lesquelles les conditions de charge deviennent de plus en plus sévères avec le temps les durées de vie prédites par le loi de Miner sont plus longues que celles calculées au départ de la vitesse de la propagation de la félure, le cas contraire étant trouvé pour les expériences dans lesquelles les conditions de charge deviennent progressivement moins sévères avec le temps. Les résultats expérimentaux sont en accord avec ceux donnés. Les effets de changement dans les paramètres structuraux ρ, E et ΔF^* et des variations de conditions expérimentales sur la précision des estimations de durée de vie sont discutés.

Zusammenfassung

Die Anwendbarkeit des Miner'schen Gesetzes der Kumulativschädigung zur Vorraussage der Lebensdauer bei Versuchen mit komplexer Belastungsvorgeschichte wird untersucht. Die durch $\Sigma[t_i/(t_b)_i] = 1$ gegebene Lebensdauer wird mit den aus dem Ausdruck von Prevorsek und Lyons für die Bruchdauer unter der Annahme, dass diese näherungsweise der Zeit zur Ausbildung eines instabilen Risses gleichkommt, berechneten verglichen. Bei Versuchen, bei welchen die Beanspruchungsbedingungen mit der Dauer zunehmend härtere werden, ist die Lebensdauer nach dem Miner'schen Gesetz länger als sich aus der Risswachstumsgeschwindigkeit berechnen lässt, während das Entgegengesetzte für Versuche mit zeitlich abnehmender Härte der Beanspruchungsbedingungen gilt. Verfügbare Versuchsdaten stimmen damit überein. Der Einfluss der Änderung der Strukturparameter ρ, E und ΔF^* sowie der Veränderung der Versuchsbedingungen auf die Genauigkeit der Lebensdauerbestimmungen wird diskutiert.

Received September 12, 1966
Prod. No. 1506